ON THE CONDITIONS OF EXISTENCE OF DECAYING SOLUTIONS OF THE TWO-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY FOR A SEMI-INFINITE STRIP

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The two-dimensional problem of a semi-infinite strip with the longitudinal sides free of stress and with various end conditions is considered. It is shown under what conditions decaying solutions of the problem exist. The necessity of these conditions arose with the exact formulation of the bending problem of a strip [1]. According to this theory, obtaining of the stress distributions on an end of the strip is equivalent to the solution of the problem of plane deformation.

The question of the conditions under which decaying solutions exist was raised by Prokopov [2]. However, he applied an insufficiently justified method of analogy, which, although it gave correct results in the case when the end is subjected to longitudinal displacement and tangential stress, lead to erroneous results when applied to the case of normal stress and transverse displacement at the end.

1. We shall take the x-axis along the center line of the semi-infinite strip and the y-axis along its end, and assume that the edges at $y = \pm 1$ and the end at infinity are free from stress. We shall consider three problems corresponding to the following conditions at the end $x = 0$

$$\sigma_x(0, y) = f_1(y), \qquad \tau_{xy}(0, y) = f_2(y) \quad \text{(problem 1)} \tag{1.1}$$

$$2\mu u(0, y) = f_1(y), \qquad \tau_{xy}(0, y) = f_2(y) \quad \text{(problem 2)} \tag{1.2}$$

$$\sigma_x(0, y) = f_1(y), \qquad 2\mu v(0, y) = f_2(y) \quad \text{(problem 3)} \tag{1.3}$$

where μ is the shearing modulus.

The boundary conditions at $y = \pm 1$ for all three problems are

$$\sigma_y(x, \pm 1) = 0, \qquad \tau_{xy}(x, \pm 1) = 0 \tag{1.4}$$

We shall consider the cases of skew-symmetrical and symmetrical deformations separately. In the first case $f_1(y)$ and $f_2(y)$ are odd and even functions of y, respectively. In the second case $f_1(y)$ is even and $f_2(y)$ is odd.

As the strip is in equilibrium, then for any stress distribution at the end at $x = 0$ the following conditions must be satisfied:

for the skew-symmetric deformation

$$\int_0^1 \sigma_x(0, y)\, y\, dy = 0, \qquad \int_0^1 \tau_{xy}(0, y)\, dy = 0 \tag{1.5}$$

for the symmetric deformation

$$\int_0^1 \sigma_x(0, y)\, dy = 0 \tag{1.6}$$

Consequently, the stresses at the end $x = 0$ are statically equivalent to zero, irrespectively, whether their distributions are known or not. According to the Saint-Venant's principle the stresses are decaying in the x-direction in all three problems. It remains to show what conditions must be satisfied by the end distributions so that the displacements are also decaying.

2. Let the solution for the two-dimensional problem be expressed in terms of the biharmonic Airy function

$$\Phi(x, y) = \sum_{k=1}^{\infty} a_k e^{-u_k x} F_k(y) \tag{2.1}$$

where $F_k(y)$ is the Papkovich function [3] satisfying the fourth order differential equation

$$F_k^{IV}(y) + 2u_k^2 F_k''(y) + u_k^4 F_k(y) = 0 \tag{2.2}$$

and the boundary conditions

$$F_k(\pm 1) = 0, \quad F_k'(\pm 1) = 0 \tag{2.3}$$

For the case of skew-symmetric deformations in (2.1), odd functions $F_k(y)$ have to be considered. These have the form

$$F_k(y) = u_k \cos u_k \sin u_k y - u_k y \cos u_k y \sin u_k \tag{2.4}$$

where u_k are the roots of Equation

$$\sin 2u - 2u = 0 \tag{2.5}$$

In the case of symmetric deformations, $F_k(y)$ are even functions of the form

$$F_k(y) = u_k \sin u_k \cos u_k y - u_k y \sin u_k y \cos u_k \tag{2.6}$$

where u_k are the roots of Equation

$$\sin 2u + 2u = 0 \tag{2.7}$$

Equation (2.1) is summed over the roots of Equation (2.5) (or (2.7)), the real parts of which are positive. The functions $F_k(y)$ satisfy the generalized orthogonality conditions introduced by Papkovich

$$\int_0^1 [F_k''(y) F_s''(y) - u_k^2 u_s^2 F_k(y) F_s(y)]\, dy = 0 \qquad (k \neq s) \tag{2.8}$$

The stresses corresponding to the stress function (2.1) have the form

$$\sigma_x = \sum_{k=1}^{\infty} a_k e^{-u_k x} F_k''(y), \qquad \sigma_y = \sum_{k=1}^{\infty} a_k u_k^2 e^{-u_k x} F_k(y)$$

$$\tau_{xy} = \sum_{k=1}^{\infty} a_k u_k e^{-u_k x} F_k'(y) \tag{2.9}$$

Using the Love's method [4] we obtain the expressions for the displacements

$$2\mu u = \sum_{k=1}^{\infty} a_k e^{-u_k x}\Big[-(1-\sigma)\frac{1}{u_k}F_k''(y) + \sigma u_k F_k(y)\Big] + u_0 \tag{2.10}$$

$$2\mu v = \sum_{k=1}^{\infty} a_k e^{-u_k x}\Big[-(1-\sigma)\frac{1}{u_k^2}F_k'''(\mu) - (2-\sigma)F_k'(y)\Big] + v_0$$

where σ is Poisson's ratio. The quantities u_0 and v_0 are linear functions of the coordinates and represent displacements of the strip as a rigid body. For the case of skew-symmetric deformations we must take

$$u_0 = -ay, \qquad v_0 = ax + b \tag{2.11}$$

and for symmetric deformations

$$u_0 = c, \qquad v_0 = 0 \tag{2.12}$$

As mentioned above, the series (2.9) and (2.10) are summed over the roots of Equations (2.5) and (2.7), the real parts of which are positive. Therefore the stresses given by (2.9) are of decaying character. If Expressions (2.10) are free of the terms u_0 and v_0, then the displacements are also of decaying character.

3. We shall consider the first problem corresponding to the boundary conditions (1.1). For the full solution of the problem it is necessary to obtain the values of a_k. The question of the existence of decaying solutions can, however, be answered without the knowledge of the coefficients a_k.

In this problem no conditions were imposed on the displacements; they are determined with accuracy up to displacements of the strip considered as an absolutely rigid body. If such displacements are not admitted, i.e. $u_0 = v_0 = 0$, then the resulting solutions for the displacements are decaying in the x-direction.

In this manner, the conditions

$$\int_0^1 f_1(y)\, y\, dy = 0, \qquad \int_0^1 f_2(y)\, dy = 0 \tag{3.1}$$

for the skew-symmetric deformation, and

$$\int_0^1 f_1(y)\, dy = 0 \tag{3.2}$$

for the symmetric deformation are necessary and sufficient conditions for the existence of decaying solutions of the first problem.

4. We shall now consider the second problem corresonding to boundary conditions (1.2). We shall treat first the case of skew-symmetric deformation. From (1.5) we obtain one condition which must be satisfied by the function $f_2(y)$

$$\int_0^1 f_2(y)\, dy = 0 \tag{4.1}$$

From conditions (1.2), taking into consideration (2.9) and (2.10), we have

$$\sum_{k=1}^{\infty} a_k\Big[-(1-\sigma)\frac{1}{u_k}F_k''(y) + \sigma u_k F_k(y)\Big] - ay = f_1(y)$$

$$\sum_{k=1}^{\infty} a_k u_k F_k'(y) = f_2(y) \tag{4.2}$$

Introducing notation

$$\psi_1(y) = \frac{\sigma}{1-\sigma}\psi_2(y) - \frac{1}{1-\sigma}f_1(y), \qquad \psi_2(y) = \int_0^y f_2(y)\, dy \tag{4.3}$$

Equations (4.2) are transformed into

$$\sum_{k=1}^{\infty} a_k \frac{1}{u_k} F_k''(y) + \frac{1}{1-\sigma} ay = \psi_1(y), \quad \sum_{k=1}^{\infty} a_k u_k F_k(y) = \psi_2(y) \tag{4.4}$$

Application of the generalized orthogonality conditions (2.8) to Equation (4.4) allow for the coefficients a_k to be determined. Accordingly we obtain

$$a_k = \frac{1}{2u_k^3 \sin^4 u_k} \int_0^1 [F_k''(y)\psi_1(y) - u_k^2 F_k(y)\psi_2(y)]\, dy \tag{4.5}$$

The constant a can also be determined from (4.4). This is done by multiplying the first relationship by y and integrating it with respect to y from $y = 0$ to $y = 1$. Remembering that

$$\int_0^1 F_k''(y)\, y\, dy = 0$$

we have

$$\frac{1}{3(1-\sigma)} a = \int_0^1 \psi_1(y)\, y\, dy$$

Substituting for $\psi_1(y)$, and integrating by parts we obtain

$$a = -3\left[\int_0^1 f_1(y)\, y\, dy + \frac{\sigma}{2}\int_0^1 f_2(y)\, y^2\, dy\right] \tag{4.6}$$

It was noted above that if in (2.10) the quantities u_0 and v_0 vanish, then the solutions for the displacements are decaying. In the case under consideration no conditions were imposed on $v(x, y)$. Consequently, it is necessary to set $v_0 = 0$.

The value of a is given by (4.6). If the condition

$$\int_0^1 f_1(y)\, y\, dy + \frac{\sigma}{2}\int_0^1 f_2(y)\, y^2\, dy = 0 \tag{4.7}$$

is satisfied, then a vanishes and from (2.11) it follows that $u_0 = 0$.

Therefore, if condition (4.7) is satisfied the displacements for this problem are of decaying character.

Condition (4.7) can be established in another manner. We assume that the solution corresponding to the boundary conditions $f_1(y)$ and $f_2(y)$ are decaying. Then it is necessary to set the quantities u_0 and v_0 in (2.10) and a in (4.2) are equal to zero. We shall find the conditions which must be satisfied by the mutually independent functions $f_1(y)$ and $f_2(y)$ so that the representations (4.2) are simultaneously valid in the region $0 \leqslant y < 1$ (with $a = 0$).

The possibility of similar developments for two functions (*) for one

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*) Note that in the paper [5] in the development of the conditions ensuring the possibility of expansions (1.7) an error is commited in the evaluation of the integral (2.16); the pole of first order at the point $z = 0$ was not taken into account. Correcting for this error we obtain instaed of (2.21)

$$\lim_{n\to\infty} A_n = 2\,[f_1(y) - f_1(1) + (1-y)\, f_1'(0) + {}^1/_4\,(1-y^2)\,(f_1'(1) - f_1'(0))]$$

and instead of (2.24)

$$V_1(y) = f_1(y) - f_1(1) + (1-y)\, f_1'(0) + {}^1/_4\,(1-y^2)\,(f_1'(1) - f_1'(0))$$

In order that the sum of the series $V_1(y)$ equals $f_1(y)$ it is sufficient to satisfy the conditions: $f_1(1) = 0, \quad f_1'(0) = 0, \; f_1'(1) = 0.$

In the paper [5] only the first two conditions were given, the second of which is automatically satisfied by an even continuous function with continuous first derivative over the interval $(-1, +1)$.

particular case was investigated by Grinberg [5]. We shall develop a similar procedure for the problem at hand.

The quantities u_k are the roots of Equation (2.5) with positive real parts. In the complex plane u, Equation (2.5) has an infinite plurality of quartets of simple complex roots, $u_k, \bar{u}_k, -\bar{u}_k, -u_k$ $(k = 1, 2, \ldots)$ and roots of third order $u = 0$.

Taking into account the fact that in series constituting (4.2), the sum of terms based on roots with positive real part equals the sum of roots with negative real parts, we shall extend the summation in (4.2) over all obtainable nonzero roots of Equation (2.5) and multiply the sums by a coefficient of $\frac{1}{2}$. We shall also replace the second of Equations (4.2) by one integrated with respect to y from 0 to y.

This results in the following equations

$$\lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{4n} a_k \left[-(1-\sigma) \frac{1}{u_k} F_k''(y) + \sigma u_k^2 F_k(y) \right] = f_1(y) \tag{4.8}$$

$$\lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{4n} a_k u_k F_k(y) = \psi_2(y)$$

Where different roots of Equation (2.5) are identified by different subscript value. We shall elaborate now on the conditions which must be satisfied by the functions $f_1(y)$ and $f_2(y)$ in order that the expansions (4.8) can exist simultaneously. We denote the left-hand parts of Equations (4.8) by $A(y)$ and $B(y)$, respectively. Substituting for a_k from (4.5) and $\psi_2(y)$ from (4.3) we obtain expressions for $A(y)$ and $B(y)$ in the form

$$A(y) = T_{f_1}^{(1)}(y) - T_{f_1}^{(2)}(y) - \frac{1}{1-\sigma} T_{f_1}^{(3)}(y) - \frac{1}{1-\sigma} T_{f_1}^{(4)}(y) -$$

$$- \sigma T_{\psi_2}^{(1)}(y) + T_{\psi_2}^{(2)}(y) + \frac{\sigma}{1-\sigma} T_{\psi_2}^{(3)}(y) - \frac{1}{1-\sigma} T_{\psi_2}^{(4)}(y) \tag{4.9}$$

$$B(y) = \frac{1}{1-\sigma} \left(- T_{f_1}^{(3)}(y) + T_{f_1}^{(4)}(y) + \sigma T_{\psi_2}^{(3)}(y) - T_{\psi_2}^{(4)}(y) \right)$$

where

$$T_\omega^{(1)}(y) = \lim_{n\to\infty} \sum_{k=1}^{4n} \frac{1}{4u_k^4 \sin^4 u_k} \int_0^1 \omega(x) F_k^*(x) F_k^*(y) dx$$

$$T_\omega^{(2)}(y) = \lim_{n\to\infty} \sum_{k=1}^{4n} \frac{1}{4u_k^4 \sin^4 u_k} \int_0^1 \omega(x) u_k^2 F_k(x) F_k^*(y) dx \tag{4.10}$$

$$T_\omega^{(3)}(y) = \lim_{n\to\infty} \sum_{k=1}^{4n} \frac{1}{4u_k^4 \sin^4 u_k} \int_0^1 \omega(x) u_k^2 F_k^*(x) F_k(y) dx$$

$$T_\omega^{(4)}(y) = \lim_{n\to\infty} \sum_{k=1}^{4n} \frac{1}{4u_k^4 \sin^4 u_k} \int_0^1 \omega(x) u_k^4 F_k(x) F_k(y) dx$$

Where $\omega(x)$ is an arbitrary odd function

$$F_k^*(x) = F_k''(x) + u_k^2 F_k(x) = 2u_k^2 \sin u_k \sin u_k y \tag{4.11}$$

We shall consider the evaluation of $T_\omega^{(1)}(y)$. Rearrangement of the order of terms gives

$$T_\omega^{(1)}(y) = \lim_{n\to\infty} \int_0^1 \omega(x) \sum_{k=1}^{\infty} \frac{F_k^*(x) F_k^*(y)}{4u_k^4 \sin^4 u_k} dx \tag{4.12}$$

Denoting the sum under the integral by $C_n(x,y)$ and introducing the function $\varphi(z) = \sin 2z - 2z$, where $\varphi'(z) = -4\sin^2 z$, and substituting for $F_k^*(x)$ and $F_k^*(y)$ from (4.11) we obtain

$$C_n(x, y) = -\sum_{k=1}^{4n} \frac{4\sin u_k y \sin u_k x}{\varphi'(u_k)} \tag{4.13}$$

Consider the line integral

$$J_n(x, y) = -\frac{1}{2\pi i}\int_{R_n} \frac{4\sin zy \sin zx}{\varphi(z)}\, dz \tag{4.14}$$

evaluated along a circle of radius R_n with its center at the origin of coordinates z. The radius R_n is selected in such a way that the circle encloses $4n$ complex roots of Equation $\varphi(z) = 0$ with the condition that differences between R_n and $|u_{4n}|$ and $|u_{4n+1}|$ are finite. When computing the residues of $T_\omega^{(i)}(y)$ $(i = 2, 3, 4)$ it is necessary to assume that the difference between R_n and the roots of Equation $\sin z = 0$ i.e. $z_s = s\pi$ $(s = 1, 2, \ldots)$ are also finite.

The integral (4.14) is equal to the sum of all residues of the integrand at the singular points. The singular points of the integrand coincide with the roots of the function $\varphi(z)$. Within the circle of radius R_n the function $\varphi(z)$ has $4n$ simple roots $z = u_k$ and one third order root $z = 0$. Correspondingly, $J_n(x,y)$ is equal to the sum of the residues of the integrand at the simple poles $z = u_k$ $(k = 1, 2, \ldots, 4n)$ and the residue at the third order pole at $z = 0$. But the sum of the residues at the simple poles is $C_n(x,y)$ and that for the third order pole $z = 0$ is $3xy$. In this manner we obtain

$$J_n(x, y) = C_n(x, y) + 3xy \tag{4.15}$$

When $R_n \to \infty$, with $0 \leqslant x \leqslant 1$, $0 \leqslant y < 1$, it follows from Jordan's Lemma that $J_n(x,y)$ tends uniformly to zero with x.

From (4.15) it follows that

$$\lim_{n\to\infty} C_n(x, y) = -3xy$$

where $C_n(x,y)$ tends to its limiting value for $n \to \infty$, uniformly with x for constant values of y. Therefore, for any function $\omega(x)$ absolutely integrable over the interval $0 \leqslant x \leqslant 1$ from (4.12) we have

$$T_\omega^{(1)}(y) = -3y\int_0^1 \omega(x)\, x\, dx \tag{4.16}$$

We shall not consider the evaluation of $T_\omega^{(i)}(y)$ $(i = 2, 3, 4)$ in detail. It can be done in basically the same manner as for $T_\omega^{(1)}(y)$. It is only necessary for the rearrangement of the orders of summation to carry out the integration by parts (once in the case of $T_\omega^{(2)}(y)$ and $T_\omega^{(3)}(y)$ and twice for $T_\omega^{(4)}(y)$, and take into account the residues arising from the poles $z_s = s\pi$ $(s = 1, 2, 3, \ldots)$, and corresponding to the roots of Equation $\sin z = 0$. The final results are

$$T_\omega^{(2)}(y) = T_\omega^{(3)}(y) = T_\omega^{(4)}(y) = -\omega(x) \tag{4.17}$$

From (4.9), (4.16) and (4.17) it follows that

$$A(y) = -3y\left[\int_0^1 f_1(x)\, x\, dx + {}^1/_2\sigma\int_0^1 f_2(x)\, x^2\, dx\right] + f_1(y)$$

$$B(y) = \psi_2(y)$$

But $A(y)$ and $B(y)$ are equal to the left-hand sides of Equations (4.8), from which it follows that if the functions $f_1(y)$ and $f_2(y)$ satisfy previously established conditions (4.7) then the series in Equations (4.8) represent the functions $f_1(y)$ and $\psi_2(y)$ for $0 \leqslant y < 1$. This proves that if the functions $f_1(y)$ and $f_2(y)$ of the second problem (1.2) with skew-symmetric deformations satisfy conditions (4.1) and (4.7), then there exist decaying solutions of the problem. Conditions (4.1) arise from equilibrium requirements and (4.7) is the condition which must be satisfied in order that the solution be representable in terms of series of Papkovich functions in the class of decaying functions.

We shall note that for all remaining cases the conditions for existence of decaying solutions shall be derived using both methods employed above. We shall agree, however, to lean more on the first simple method, which is based on the determination of boundary conditions for the nondecaying components of the expressions for the displacements and equating them to zero.

We shall now consider the symmetrical deformation of the strip under conditions (1.2).

Equilibrium conditions (1.6) do not impose any conditions on the function $f_1(y)$ and $f_2(y)$. Taking into account Equations (2.9), (2.10) and (2.12), the boundary conditions (1.2) give

$$\sum_{k=1}^{\infty} a_k\left[-(1-\sigma)\frac{1}{u_k}F_k''(y)+\sigma u_k F_k(y)\right]+c=f_1(y)$$

$$\sum_{k=1}^{\infty} a_k u_k F_k'(y)=f_2(y) \qquad (4.18)$$

where $F_k(y)$ are the even Papkovich functions (2.6). The properties of Papkovich functions allow for the determination of the coefficients a_k ($k = 1, 2,\ldots$) and the quantity c from the system (4.18). However for our purpose it is sufficient to determine only the quantity c.

Integrating with respect to y the first of (4.18) and the second of (4.18) premultiplied by y, over the interval 0 to 1, one readily obtains

$$c=\int_0^1 f_1(y)\,dy+\sigma\int_0^1 f_2(y)\,y\,dy \qquad (4.19)$$

From (4.19) and (2.12) it follows, that if the condition

$$\int_0^1 f_1(y)\,dy+\sigma\int_0^1 f_2(y)\,y\,dy=0 \qquad (4.20)$$

is satisfied, then the quantity c, and consequently u_0, vanish.

Therefore if the functions $f_1(y)$ and $f_2(y)$ in the second problem with symmetrical deformations satisfy condition (4.20), there exist decaying solutions of the problem.

5. We shall now consider the third problem corresponding to the boundary conditions (1.3). In the case of skew-symmetric deformation the equilibrium conditions (1.5) impose one condition on the function $f_1(y)$. It has the form

$$\int_0^1 f_1(y)\,y\,dy=0 \qquad (5.1)$$

From the boundary conditions (1.3) it follows that

$$\sum_{k=1}^{\infty} a_k F_k''(y)=f_1(y)$$

$$\sum_{k=1}^{\infty} a_k\left[-(2-\sigma)F_k'(y)-(1-\sigma)\frac{1}{u_k^2}F_k'''(y)\right]+b=f_2(y) \qquad (5.2)$$

The quantity b is determined from (5.2). For this purpose we integrate Equations (5.2) with respect to y (the first premultiplied by y^3 and the second by y^2) over the interval 0 to 1. From the results, it is easy to obtain

$$b = \frac{1}{2}\left[(2-\sigma)\int_0^1 f_1(y)\,y^3\,dy + 3\int_0^1 f_2(y)\,(1-y^2)\,dy\right]$$

If the condition

$$(2-\sigma)\int_0^1 f_1(y)\,y^3\,dy + 3\int_0^1 f_2(y)\,(1-y^2)\,dy = 0 \tag{5.3}$$

is satisfied then $b = 0$. No conditions can be imposed on the displacement $u(0,y)$. It is necessary, therefore, to set $u_0 = 0$, i.e. $a = 0$. Then it follows from (2.11) that $v_0 = 0$.

This shows that in the third problem (1.3) with skew-symmetric deformation, if the functions $f_1(y)$ and $f_2(y)$ satisfy the conditions (5.1) and (5.3) then decaying solutions exist.

We shall now consider the symmetrical deformations with the boundary conditions (1.3).

From equilibrium conditions (1.6) we obtain the condition to be satisfied by $f_1(y)$

$$\int_0^1 f_1(y)\,dy = 0 \tag{5.4}$$

No conditions were imposed on $u(0, y)$ in this case. It is, therefore, necessary to set $u_0 = 0$ in (2.10). Therefore, in the third problem (1.3) with symmetric deformations if condition (5.4) is imposed on the function $f_1(y)$, then the problem has decaying solutions.

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